Impacts of disasters on coastal environments

Angela Celeste TARAMASSO PhD

angela.celeste.taramasso@unige.it

Linear wave theory.

The most elementary wave theory is the *small-amplitude* or *linear wave theory*. This theory, developed by Airy (1845), is easy to apply, and gives a **reasonable approximation of wave characteristics for a wide range of wave parameters**.

A more complete theoretical description of waves may be obtained as the sum of many successive approximations, where each additional term in the series is a correction to preceding terms. For some situations, waves are better described by these higher-order theories, which are usually referred to as *finite-amplitude wave theories* (Mei 1991, Dean and Dalrymple 1991).

Although there <u>are limitations to its applicability</u>, linear theory can still be useful provided the assumptions made in developing this simple theory are not grossly violated.

The **assumptions** made in developing the linear wave theory are:

- the fluid is homogeneous and incompressible; therefore, the density ρ is a constant;
- surface tension can be neglected;
- coriolis effect due to the earth's rotation can be neglected;
- pressure at the free surface is uniform and constant;
- the fluid is ideal or inviscid (lacks viscosity);
- the particular wave being considered does not interact with any other water motions. The flow is irrotational so that water particles do not rotate (only normal forces are important and shearing forces are negligible);
- the bed is a horizontal, fixed, impermeable boundary, which implies that the vertical velocity at the bed is zero;
- the wave amplitude is small and the waveform is invariant in time and space;
- waves are plane or long-crested (two-dimensional).

The assumption of irrotationality stated as the sixth assumption above allows the use of a mathematical function termed the *velocity potential* Φ. The velocity potential is a scalar function whose gradient (i.e., the rate of change of Φ relative to the x-and z-coordinates in two dimensions where x = horizontal, z = vertical) at any point in fluid is the velocity vector. Thus,

$u = \delta\Phi/\delta x$

is the fluid velocity in the x-direction, and

$w = \delta\Phi/\delta z$

is the fluid velocity in the z-direction.

Φ has the units of length squared divided by time.

Consequently, if Φ(x, z, t) is known over the flow field, then fluid particle velocity components *u* and *w* can be found.

The incompressible assumption implies that there is another mathematical function termed the *stream function* Ψ. Some wave theories are formulated in terms of the stream function Ψ, which is orthogonal to the potential function Φ.

Lines of constant values of the potential function (equipotential lines) and lines of constant values of the stream function are mutually perpendicular or orthogonal. Consequently, if Φ is known, Ψ can be found, or vice versa, using the equations

$$\delta \Phi / \delta x = \delta \Psi / \delta z$$

$$\delta \Phi / \delta z = - \delta \Psi / \delta x$$

termed the *Cauchy-Riemann conditions* (Whitham 1974; Milne-Thompson 1976).

Both Φ and Ψ satisfy the *Laplace equation* which governs the flow of an *ideal fluid* (inviscid and incompressible fluid).
Thus, under the assumptions outlined above, the Laplace equation governs the flow beneath waves. The Laplace equation in two dimensions with x = horizontal, and z = vertical axes in terms of velocity potential Φ is given by

$$\delta^2\Phi/\delta x^2 + \delta^2\Phi/\delta z^2 = 0$$

In terms of the stream function, Ψ, Laplace's equation becomes

$$\delta^2\Psi/\delta x^2 + \delta^2\Psi/\delta z^2 = 0$$

The symbol η denotes the displacement of the water surface relative to the SWL and is a function of x and time t. At the wave crest, η is equal to the amplitude of the wave a, or one-half the wave height $H/2$.

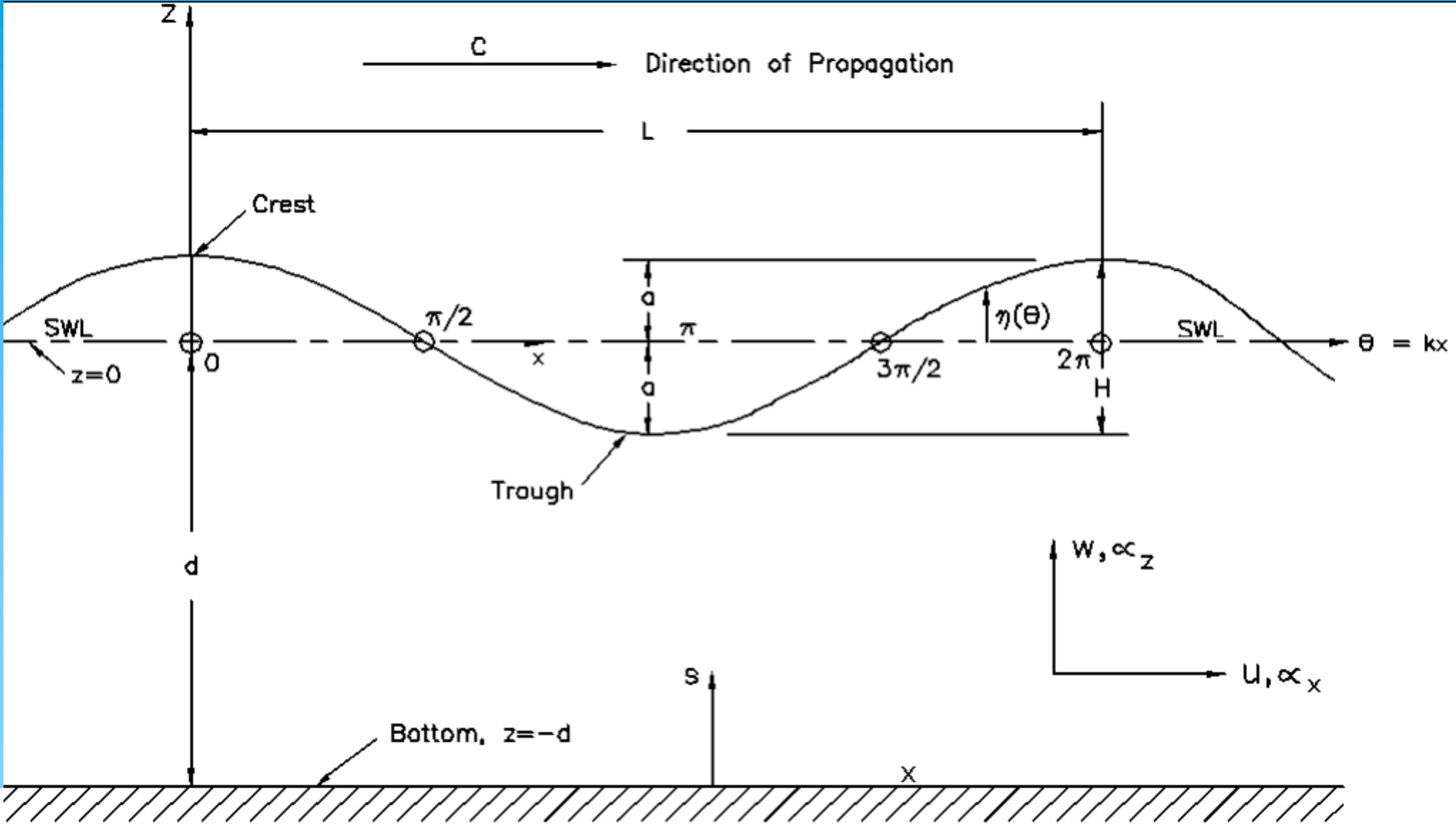
The speed at which a wave form propagates is termed the *phase velocity* or *wave celerity* C. Since the distance traveled by a wave during one wave period is equal to one wavelength, wave celerity can be related to the wave period and length by

$$C = L/T$$

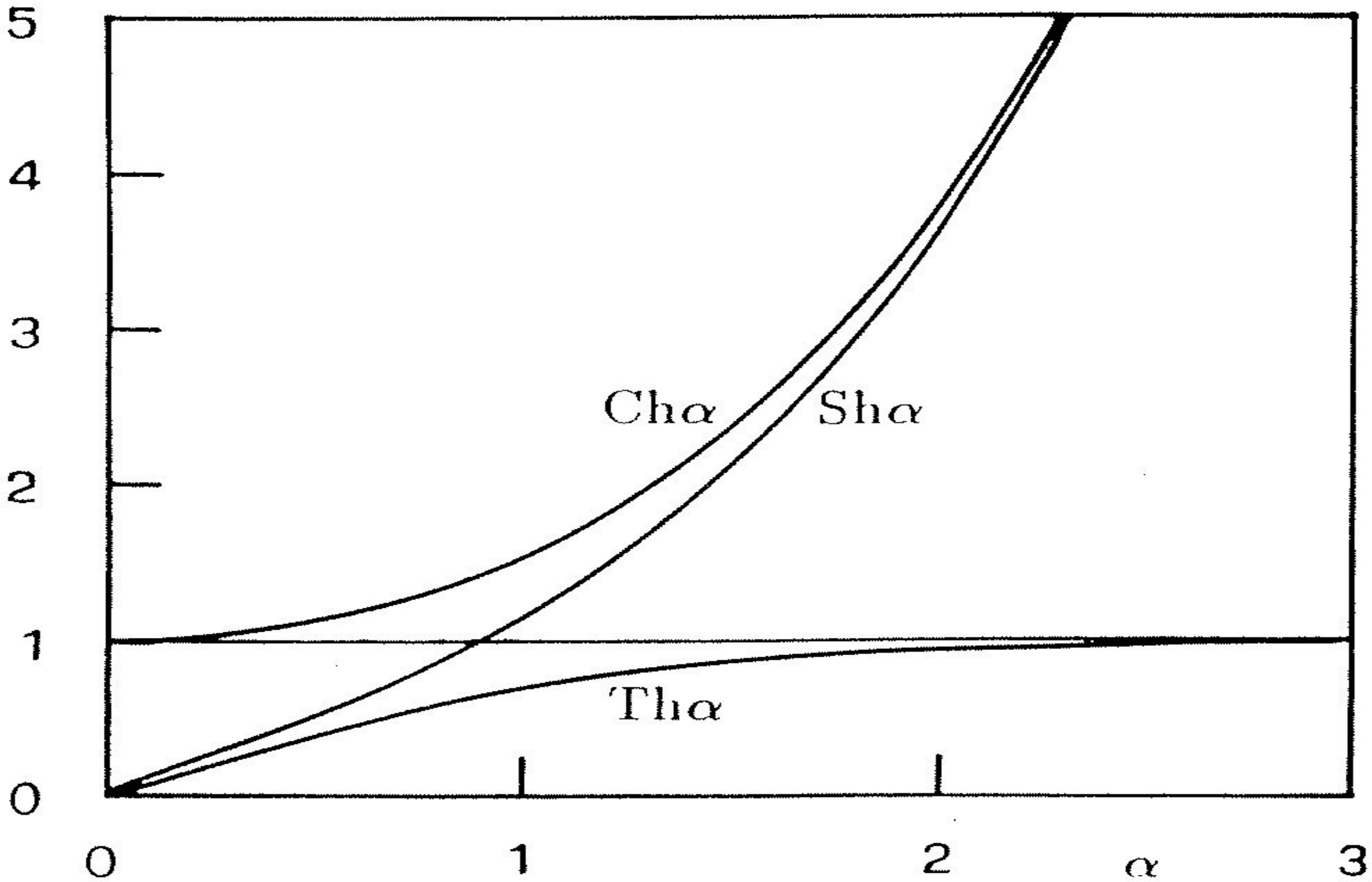
An expression relating wave celerity to wavelength and water depth is given by

$$C = \sqrt{[gLTh(2\pi d/L)]/(2\pi)}$$

known as ***dispersion relation*** since it indicates that waves with different periods travel at different speeds. For a situation where more than one wave is present, the longer period wave will travel faster.



Z
C
Direction of Propagation
L
Crest
a
a
π/2
π
η(θ)
SWL
SWL
z=0
0
x
3π/2
2π
H
θ = kx
Trough
d
W, ∝z
U, ∝x
S
Bottom, z=−d
X



$$\mathrm{Ch}\alpha = [\exp\alpha + \exp(-\alpha)]/2$$
$$\mathrm{Sh}\alpha = [\exp\alpha - \exp(-\alpha)]/2$$
$$\mathrm{Th}\alpha = \mathrm{Sh}\alpha/\mathrm{Ch}\alpha.$$

Per $\alpha \geq \pi$ risulta:

$$\mathrm{Ch}\alpha = \mathrm{Sh}\alpha = (\exp\alpha)/2; \quad \mathrm{Th}\alpha = 1.$$

Per $\alpha \leq \pi/10$ risulta:

$$\mathrm{Sh}\alpha \sim \alpha; \quad \mathrm{Ch}\alpha \sim 1; \quad \mathrm{Th}\alpha \sim \alpha.$$

The values *$2\pi/L$* and *$2\pi/T$* are called the *wave number k* and the *wave angular frequency ω*, respectively. Then an expression for wavelength as a function of depth and wave period may be obtained as

$$L = gT^2/2\pi * Th(2\pi d/L)$$

The unknown value of L appears on both sides of the equation, *to solve it is necessary to define L_0* as the deepwater wavelength, or to utilize the following expression which is correct to within about 10 percent

$$L = gT^2/2\pi * \sqrt{Th[(2\pi)^2 d/(T^2 g)]}$$

Classification of Water waves			
Classification	d/L	kd	Th(kd)
Deep water	0,5 to ∞	π to ∞	= 1
Trasitional	0,05 to 0,5	π/10 to π	Th (kd)
Shallow water	0 to 0,05	0 to π/10	= kd

In wave force studies, the local fluid velocities and accelerations for various values of **z** and **t** during the passage of a wave must often be found. The horizontal component **u** and the vertical component **w** of the local fluid velocity are given by the following equations

$$u = (H/2)*(gT/L)*\cos\theta*[\cosh(2\pi(z+d)/L)]/\cosh(2\pi d/L)$$

$$w = (H/2)*(gT/L)*\sin\theta*[\sinh(2\pi(z+d)/L)]/\cosh(2\pi d/L)$$

These equations express the local fluid velocity components any distance $(z + d)$ above the bottom.

The velocities are periodic in both **x** and **t**. For a given value of the phase angle $\theta = (2\pi x/L - 2\pi t/T)$, the hyperbolic functions cosh and sinh, as functions of z result in an approximate exponential decay of the magnitude of velocity components with increasing distance below the free surface.

The **maximum positive horizontal velocity** occurs when $\theta = 0, 2\pi$, etc., while the maximum horizontal velocity in the negative direction occurs when $\theta = \pi, 3\pi$, etc. On the other hand, the **maximum positive vertical velocity** occurs when $\theta = \pi/2, 5\pi/2$, etc., and the maximum vertical velocity in the negative direction occurs when $\theta = 3\pi/2, 7\pi/2$, etc.

The local fluid particle accelerations are obtained by differentiating the equations of velocity with respect to t. Thus

$$a_x = \delta u/\delta t = (g\pi H/L)*\sin\theta*[\cosh(2\pi(z+d)/L)/(\cosh(2\pi d/L)]$$

$$a_z = \delta w/\delta t = (g\pi H/L)*\cos\theta*[\sinh(2\pi(z+d)/L)/(\cosh(2\pi d/L)]$$

Relative Depth	Shallow Water $\frac{d}{L} < \frac{1}{25}$	Transitional Water $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	Deep Water $\frac{d}{L} < \frac{1}{2}$
1. Wave profile	Same As >	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	< Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water particle velocity (a) Horizontal (b) Vertical	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$ $w = \frac{H\pi}{T}\left(1+\frac{z}{d}\right)\sin\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$ $w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$u = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$ $w = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
6. Water particle accelerations (a) Horizontal (b) Vertical	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$ $a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1+\frac{z}{d}\right)\cos\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$ $a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$ $a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
7. Water particle displacements (a) Horizontal (b) Vertical	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$ $\zeta = \frac{H}{2}\left(1+\frac{z}{d}\right)\cos\theta$	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$ $\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\xi = -\frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$ $\zeta = \frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
8. Subsurface pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho gz$	$p = \rho g\eta\, e^{\left(\frac{2\pi z}{L}\right)} - \rho gz$

In the following a sketch of the local fluid motion, that indicates that the fluid under the crest moves in the direction of wave propagation and returns during passage of the trough. Linear theory does not predict any net mass transport; hence, the sketch shows only an oscillatory fluid motion.

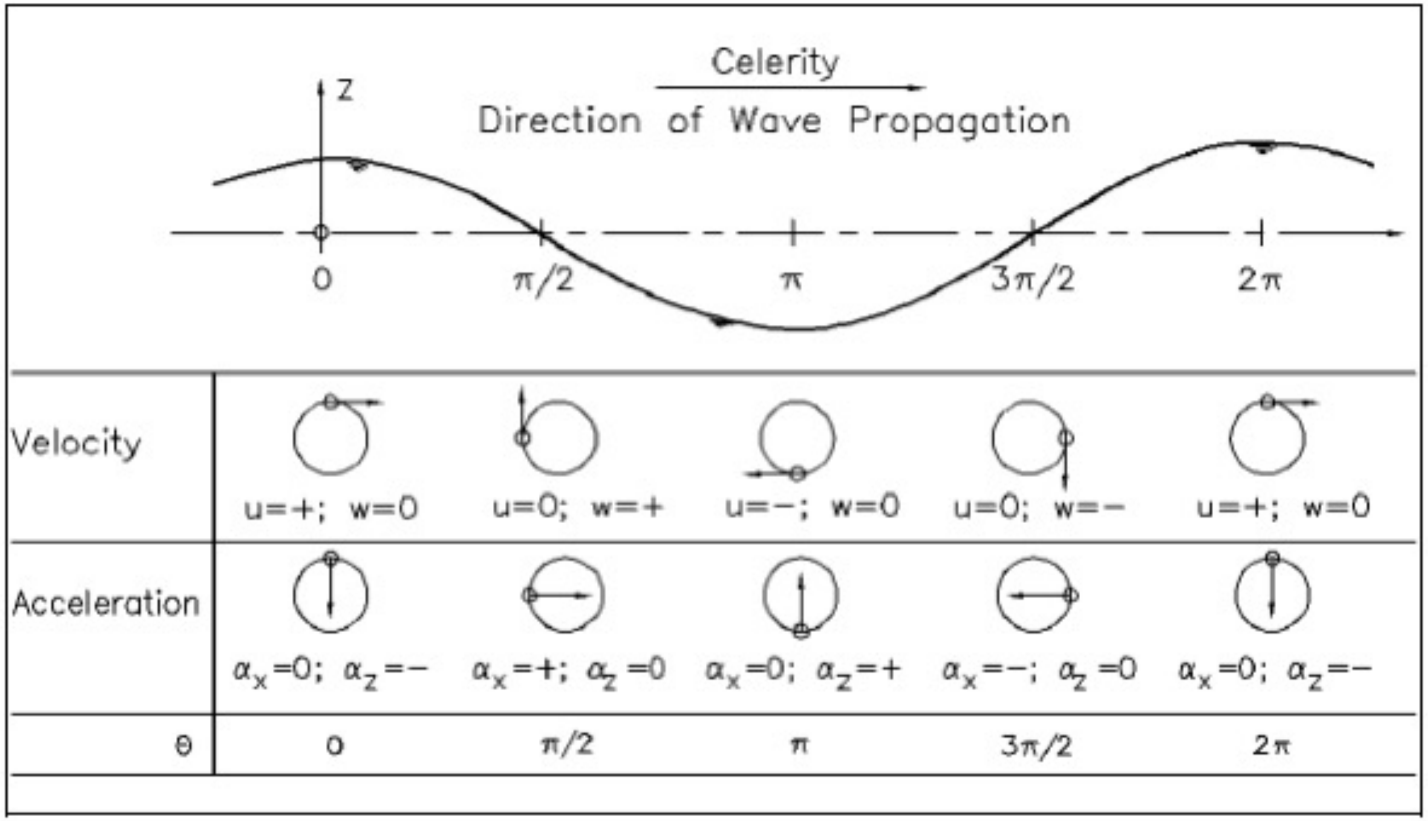


Figure II-1-2. **Local fluid velocities and accelerations**

source Coastal Engineering Manual

The following figure depicts profiles of the surface elevation, particle velocities, and accelerations by the linear wave theory.

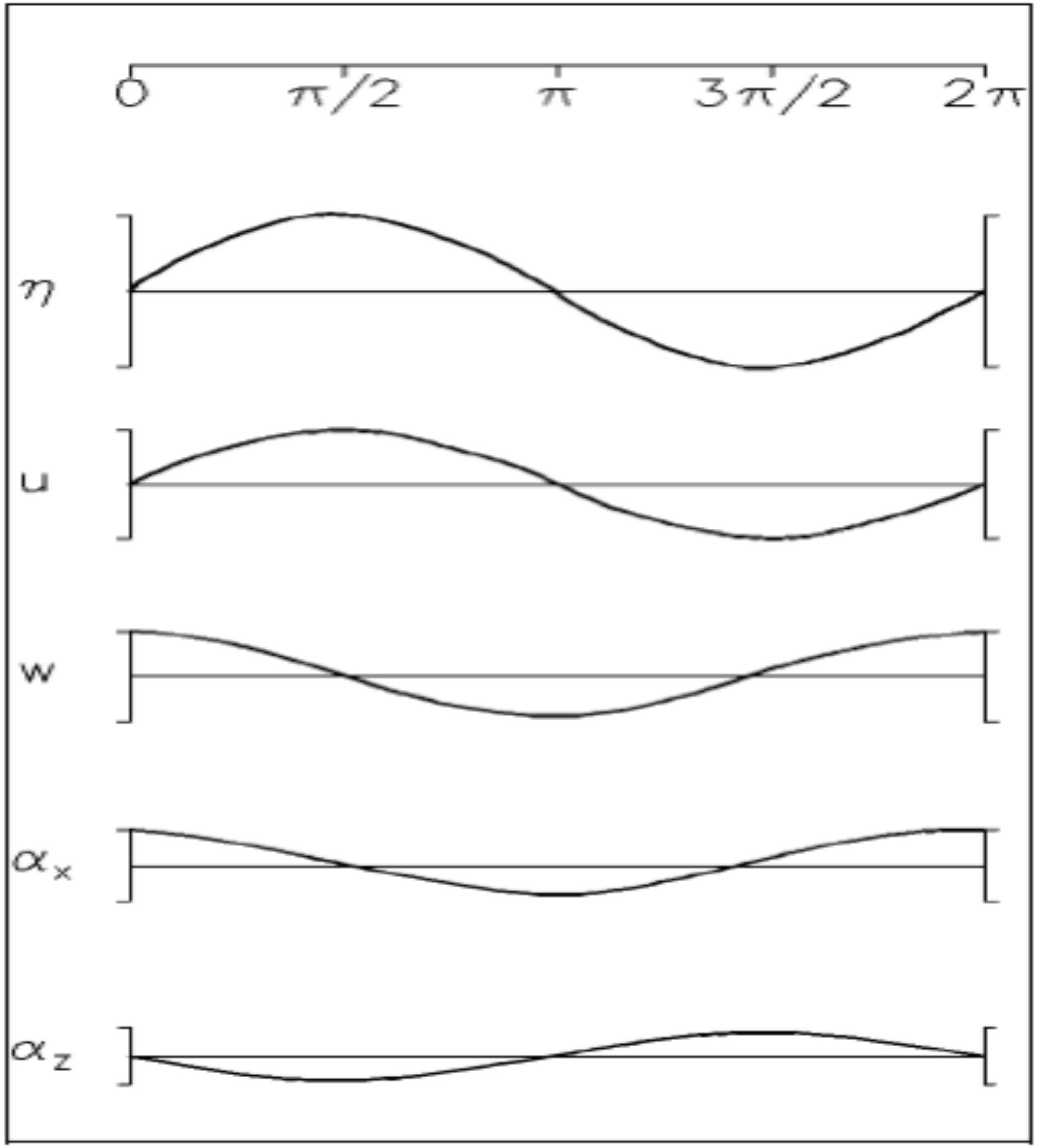


Figure II-1-3. Profiles of particle velocity and acceleration by Airy theory in relation to the surface elevation

source Coastal Engineering Manual

Another important aspect of linear wave theory deals with the displacement of individual water particles within the wave. Water particles generally move in elliptical paths in shallow or transitional depth water and in circular paths in deep water (see next figure). If the mean particle position is considered to be at the center of the ellipse or circle, then vertical particle displacement with respect to the mean position cannot exceed **one-half the wave height**. Thus, since the wave height is assumed to be small, the displacement of any fluid particle from its mean position must be small.

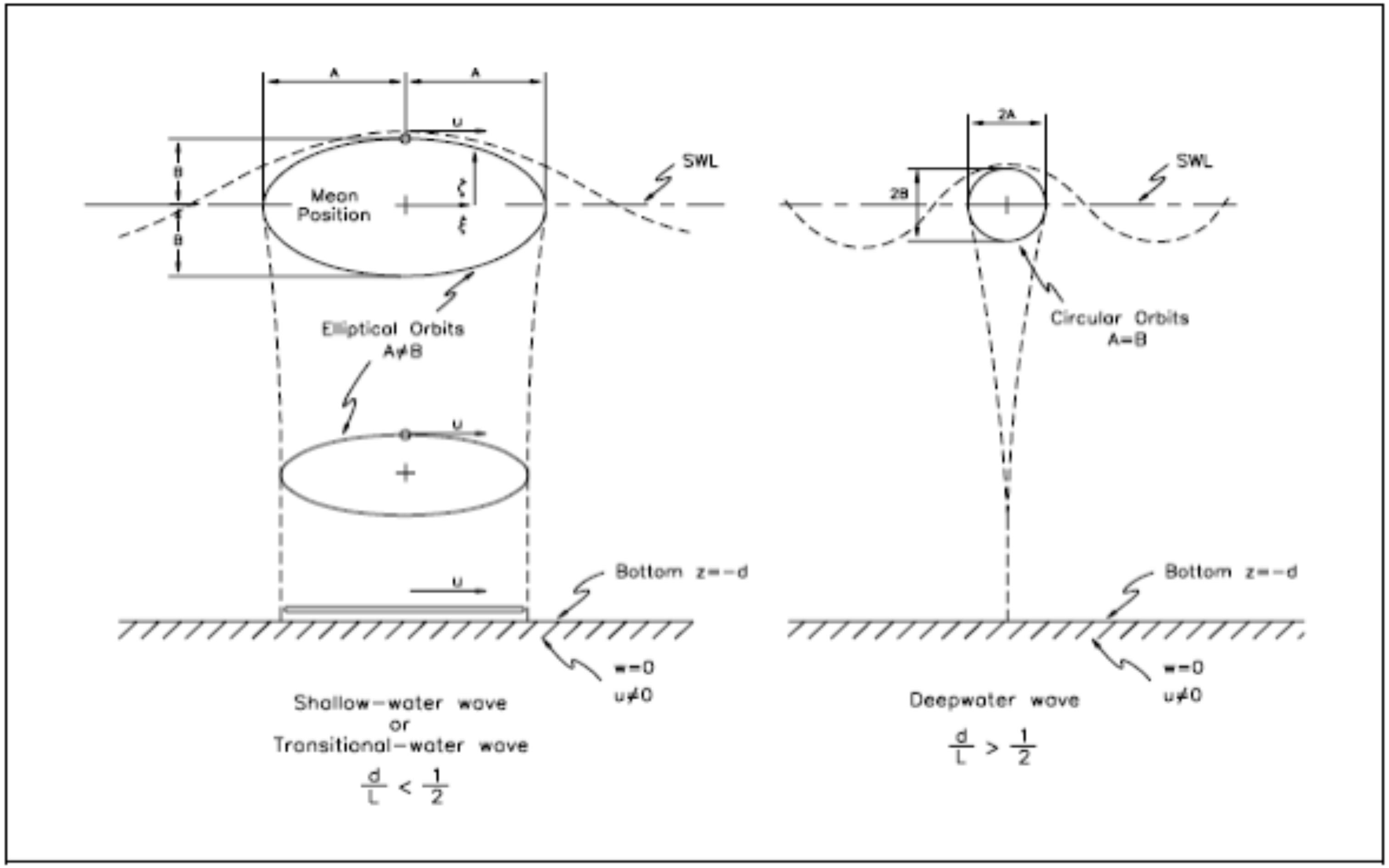


Figure II-1-4. Water particle displacements from mean position for shallow-water and deepwater waves

source Coastal Engineering Manual

Thus, in deep water, the water particle orbits are circular, and the figures show that in transitional and shallow water, the orbits are elliptical. The more shallow the water, the flatter the ellipse. The amplitude of the water particle displacement decreases exponentially with depth and in **deep water regions** becomes small relative to the wave height at a depth equal to one-half the wavelength below the free surface; i.e., when $z = L_0/2$.

For **shallow regions**, horizontal particle displacement near the bottom can be large. In fact, this is apparent in offshore regions seaward of the breaker zone where <u>wave action and turbulence lift bottom sediments into suspension</u>. The vertical displacement of water particles varies from a minimum of zero at the bottom to a maximum equal to one-half the wave height at the surface.

Subsurface pressure under a wave is the sum of two contributing components, dynamic and static pressures, and is given by

$$p = -\rho g z - p_a + \cos\theta * [\rho g H * \cosh(2\pi(z+d)/L)]/[2\cosh(2\pi d/L)]$$

where p is the total or absolute pressure, p_a is the atmospheric pressure, and ρ is the mass density of water. The first term represents a dynamic component due to acceleration, while the second term is the static component of pressure.

For convenience, the pressure is usually taken as the gauge pressure defined as

$$p = p' - p_a$$

Wave energy

The total energy of a wave system is the sum of its kinetic energy and its potential energy. The kinetic energy is that part of the total energy due to water particle velocities associated with wave motion. The **kinetic energy** per unit length of wave crest for a wave defined with the linear theory can be found from

$$E_k = 1/16 * \rho g H^2 L$$

Potential energy is that part of the energy resulting from part of the fluid mass being above the trough: the wave crest. The potential energy per unit length of wave crest for a linear wave is given by

$$E_p = 1/16 * \rho g H^2 L$$

According to the Airy theory, if the potential energy is determined relative to SWL, and all waves are propagated in the same direction, potential and kinetic energy components are equal, and the total wave energy in one wavelength per unit crest width is given by

$$E = E_k + E_p = 1/8 * \rho g H^2 L$$

Total average wave energy per unit surface area, termed the *specific energy* or *energy density*, is given by

$$\bar{E} = E/L = 1/8 * \rho g H^2$$